

COMPOSED GRAND LEBESGUE SPACES

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Abstract. In this article we introduce and investigate a new class of rearrangement invariant (r.i.) Banach function spaces, so-called Composed Grand Lebesgue Spaces (CGLS), in particular, Integral Grand Lebesgue Spaces (IGLS), which are some generalizations of known Grand Lebesgue Spaces (GLS).

We consider the fundamental functions of CGLS, calculate its Boyd's indices, obtain the norm boundedness some (regular and singular) operators in this spaces, investigate the conjugate and associate spaces, show that CGLS obeys the absolute continuous norm property etc.

Key words and phrases: Measurable spaces and functions, Grand and ordinary Lebesgue Spaces (GLS), Composed and Integral Grand Lebesgue Spaces (CGLS and IGLS), Hilbert transform and other singular and regular operators, Orlicz and other rearrangement invariant (r.i.) spaces, Fourier integrals and series, operators, equivalent norms, upper and lower estimations, Boyd's indices, dilation, conjugate and associate spaces.

Mathematics Subject Classification 2000. Primary 42Bxx, 4202; Secondary 28A78, 42B08.

1 Introduction. Definition of composed Grand Lebesgue Spaces. Simple properties

1. Grand Lebesgue Spaces.

Let (T, \mathcal{A}, μ) be some measurable space with sigma-finite non-trivial measure μ . For the measurable real valued function $f(t)$, $t \in X$, $f : T \rightarrow R$ the symbol $|f|_p = |f|_p(X, \mu)$ will denote the usually L_p Lebesgue - M.Riesz norm:

$$|f|_p = \|f\|_{L_p(X, \mu)} = \left[\int_X |f(t)|^p \mu(dt) \right]^{1/p}, \quad p \geq 1. \quad (1.1)$$

and correspondingly Lebesgue - M.Riesz spaces

$$L_p = L_p(X, \mu) = \{f : X \rightarrow R, |f|_p < \infty\}.$$

We recall in this section for reader convenience some definitions and facts from the theory of (ordinary) Grand Lebesgue Spaces (GLS) spaces.

Recently, see [8], [9], [10], [12], [13], [15], [19], [21], [22], etc. appears the so-called Grand Lebesgue Spaces $GLS = G(\psi) = G\psi = G(\psi; a, b)$, $a, b = \text{const}, a \geq 1, a < b \leq \infty$, spaces consisting on all the measurable functions $f : T \rightarrow R$ with finite norms

$$\|f\|_{G(\psi)} \stackrel{\text{def}}{=} \sup_{p \in (a, b)} [|f|_p / \psi(p)]. \quad (1.2)$$

Here $\psi(\cdot)$ is some continuous positive on the *open* interval (A, B) function such that

$$\inf_{p \in (a, b)} \psi(p) > 0, \quad \psi(p) = \infty, \quad p \notin (a, b).$$

The set of all ψ functions with support $\text{supp}(\psi) = (a, b)$ will be denoted by $\Psi(a, b)$.

This spaces are rearrangement invariant, see [4], chapter 2, and are used, for example, in the theory of probability [15], [21], [22]; theory of Partial Differential Equations [9], [13]; functional analysis [10], [12], [19], [22], [31]; theory of Fourier series [21]; theory of martingales [22]; mathematical statistics [23], [24], [25], [26], [27], [28], [29]; theory of approximation [36] etc.

We introduce and investigate some extrapolation of GLS spaces, a new class of rearrangement invariant (r.i.) Banach function spaces, so-called Composed Grand Lebesgue Spaces (CGLS), in particular, Integral Grand Lebesgue Spaces (IGLS).

We consider the fundamental functions of CGLS, calculate its Boyd's indices, obtain the norm boundedness some (regular and singular) operators in this spaces, investigate the conjugate and associate spaces, show that CGLS obeys the absolute continuous norm property etc.

WE WILL DENOTE BY $C_k = C_k(\cdot)$, $k = 1, 2, \dots$ SOME POSITIVE FINITE ESSENTIALLY AND BY C, C_0 NON-ESSENTIALLY "CONSTRUCTIVE" CONSTANTS.

2. Composed Grand Lebesgue Spaces.

Let $(X, \|\cdot\|_X)$ be a r.i. space, where X is linear subset on the space of all measurable function $(a, b) \rightarrow R$ over our measurable space (T, M, μ) with norm $\|\cdot\|_X$.

Definition.

WE WILL SAY THAT THE SPACE X WITH THE NORM $\|\cdot\|_X$ IS *Composed Grand Lebesgue Space*, BRIEFLY: $CGLS = CGLS(T, \mu; a, b; \langle \cdot \rangle) = CGLS(T, \mu; \langle \cdot \rangle)$ SPACE, IF THERE EXIST A REAL CONSTANTS $a, b; 1 \leq a < b \leq \infty$, AND SOME *rearrangement invariant norm* $\langle \cdot \rangle$ DEFINED ON THE SPACE OF A REAL FUNCTIONS DEFINED ON THE INTERVAL (a, b) , NON NECESSARY TO BE FINITE ON ALL THE FUNCTIONS SUCH THAT

$$\forall f \in X \Rightarrow \|f\|_X = \langle h(\cdot) \rangle, \quad h(p) = |f|_p. \quad (1.3)$$

Hereafter we will suppose that the norm $\|\cdot\|_X$ is order continuous.

Another approach to the problem of syntheses of the spaces L_p see in [6], [46].

Recall (see [14], chapter 4, section 3) that this means that for arbitrary sequence of a functions $g_n(p)$, $g_n(p) \in X$ the following implication holds:

$$g_n(\cdot) \downarrow 0 \implies \langle g_n \rangle \downarrow 0. \quad (1.4)$$

Note that the norm in the ordinary GLS is not order continuous.

We will write for considered CGLS spaces $(X, \|\cdot\|_X)$

$$(a, b) \stackrel{\text{def}}{=} \text{supp}(X),$$

moment support; not necessary to be uniquely defined. But we will understand as the interval (a, b) its *minimal* value.

It is obvious that arbitrary CGLS space is r.i. space.

There are many r.i. spaces satisfied the condition (1.4): exponential Orlicz's spaces, some Marzinkiewicz spaces, interpolation spaces (see [14], [42]).

An important example.

Let $Q = \text{const} \geq 1$ and let ν be some Borelian sigma-finite measure on the set (a, b) . We introduce the Integral Grand Lebesgue Space (IGLS) $G_{\mu, \nu}^{(Q)} = G_{\mu, \nu}^{(Q)}(a, b)$ as the set of all (measurable) functions $f : T \in R$ with finite norm:

$$\|f\|_Q^{(\mu, \nu)} \stackrel{\text{def}}{=} \left[\int_{[a, b]} |f|_p \nu(dp) \right]^{1/Q}. \quad (1.5)$$

Remark 1.1. If the measure ν coincides with a Dirac measure $\delta(p - p_0)$:

$$\int_a^b h(p) \nu(dp) = f(p_0),$$

then the Integral Grand Lebesgue Space (IGLS) $G_{\mu, \nu}^{(Q)}$ is equal to the Lebesgue-Riesz's space L_{p_0} .

Remark 1.2. Notice that if $\nu\{\infty\} = 0$ or $b < \infty$, that this space satisfied the condition (1.4).

Subexample. In the article of S.F.Lukomsky [20] are introduced the so-called $G(p, \alpha)$ spaces consisted on all the measurable function $f : T \rightarrow R$ with finite norm

$$\|f\|_{p,\alpha} = \left[\int_1^\infty \left(\frac{|f|_x}{x^\alpha} \right)^p \nu(dx) \right]^{1/p},$$

where ν is some Borelian measure. Lukomsky considered some applications of these spaces in the theory of Fourier series.

Astashkin in [1], [2] proved that the space $G(p, \alpha)$ in the case $T = [0, 1]$ and $\nu = m$, $m(\cdot)$ is Lebesgue measure, coincides with the Lorentz $\Lambda_p(\log^{1-p\alpha}(2/s))$ space. Therefore, both this spaces are CGLS spaces.

Some applications of the CGLS spaces in the approximation theory see in the article [30].

Theorem 1.1. The Composed Grand Lebesgue Spaces (CGLS) are complete order continue rearrangement invariant Banach functional spaces with Fatou property.

Proof. It is suffices to prove only the completeness of these spaces. In turn it is suffices to prove the order continuity of the norm in this spaces, see, e.g., [14], chapter 4, section 3.

Let $g_n = g_n(t)$ be a monotonically decreasing sequence of a functions belonging to the CGLS such that as $n \rightarrow \infty$ we have

$$g_n(t) \downarrow 0$$

μ - almost everywhere. As long as the classical Lebesgue-Riesz's space L_p obeys the order continuity of the norm, as $n \rightarrow \infty$

$$|g_n(\cdot)|_p \downarrow 0$$

for all the values p ; $p \in (a, b)$.

Since the norm $\langle \cdot \rangle$ has the order continuity of the norm, we conclude

$$\|g_n\|_X = \langle |g_n(\cdot)|_p \rangle \downarrow 0.$$

This completes the proof of theorem 1.1.

For instance, the last passing to the limit as $n \rightarrow \infty$ for the Integral Grand Lebesgue Spaces may be grounded by means of Lebesgue dominated convergence theorem.

2 Norm absolutely continuity (ACN)

Recall that the function f from the rearrangement invariant space X with the norm $\|\cdot\|_X$ has absolutely continuous norm: $f \in ACN$ in the terminology, e.g. of the

book [4], chapters 2,3, if for arbitrary decreasing sequence of measurable sets E_n such that $\mu -$ almost everywhere

$$\cap_n E_n = \emptyset$$

there holds

$$\lim_{n \rightarrow \infty} \|f \cdot I(E_n)\|_X = 0.$$

Hereafter $I(A) = I(A, t)$ denotes the indicator function for the set A :

$$I(A, t) = 1, \quad t \in A; \quad I(A, t) = 0, \quad t \notin A.$$

The whole space X has ACN property, iff arbitrary function $f, f \in X$ has ACN property.

Theorem 2.1 The CGLS space $X = CGLS(T, \mu; a, b; <>)$ has ACN property.

Proof is at the same as in theorem 1.1. Indeed, let

$$E_{n+1} \subset E_n, \quad \cap_n E_n = \emptyset,$$

and let $f \in X$; then $|f| \cdot I(E_n) \in X$ and $|f| \cdot I(E_n) \downarrow 0$ $\mu -$ a.e. Therefore $\| |f| \cdot I(E_n) \|_X \downarrow 0$, Q.E.D.

Corollary 2.1. The CGLS space $X = CGLS(T, \mu; a, b; <>)$ has *Lebesgue*, or *dominated convergence* property:

$$\forall \{f_n\} : |f_n| \leq f, \quad f, f_n \in X, \quad f_n \xrightarrow{\mu - a.e.} g \Rightarrow \|f_n - g\|_X \rightarrow 0.$$

3 Separability and reflexivity.

Theorem 3.1. Assume that the measure μ on the set A is separable; this imply by definition that the space \mathcal{A} is separable relatively a distance

$$d(B_1, B_2) \stackrel{def}{=} \mu(B_1 \setminus B_2) + \mu(B_2 \setminus B_1) = \mu(B_1 \Delta B_2), \quad B_1, B_2 \in \mathcal{A}.$$

Then the space $CGLS(T, \mu; <>)$ is separable.

Proof follows immediately from the proved order continuity of this space and from the Theorem 3 of monograph Kantorovich L.V., Akilov G.P. [14], chapter 4, section 3.

Theorem 3.2. Consider the space (IGLS) $G_{\mu, \nu}^{(Q)}$ on the set $p \in [a, b]$, where $\nu\{1\} = \nu\{\infty\} = 0$ or $1 < a < b < \infty$.

If the measure ν is purely atomic:

$$\int_a^b h(p) \nu(dp) = \sum_{k=1}^n c(k) f(p_k), \quad c(k) > 0, \quad p_k \in (a, b),$$

then the space $G_{\mu,\nu}^{(Q)}$ is reflexive.

Proof is obvious:

$$\|f\|_{G_{\mu,\nu}^{(Q)}} = \left[\sum_{k=1}^n c(k) |f|_{p_k}^Q \right]^{1/Q}$$

and follows from the reflexivity of all the spaces L_{p_k} . Note also that in considered case the space $G_{\mu,\nu}^{(Q)}$ coincides up to norm equivalence with the direct sum of L_{p_k} spaces.

Remark 3.1. The author does not know an essential generalization of this result.

4 Boyd's indices for IGLS spaces

Let in this section $T = R_+ = (0, \infty)$ with Lebesgue measure. Introduce the so-called dilation operator (more exactly, the family of operators) σ_s by the formula

$$\sigma_s[f](t) = f(t/s). \quad (4.1)$$

For arbitrary r.i. space over $T = R_+$, for example, for the space $Y = CGLS(T, \mu; a, b; < >)$ the Boyd's indices $\alpha(Y), \beta(Y)$ may be defined as follows:

$$\beta(Y) \stackrel{def}{=} \lim_{s \rightarrow \infty} \frac{\log \|\sigma_s\|_Y}{|\log s|},$$

$$\alpha(Y) \stackrel{def}{=} \lim_{s \rightarrow 0+} \frac{\log \|\sigma_s\|_Y}{|\log s|}.$$

These limits there exists and play a very important role in the theory of Fourier series and in the theory of singular integral operators in r.i. spaces, see [4], chapters 5,6.

Theorem 4.1. Let ν be Lebesgue measure:

$$\nu(A) = \int_A dp,$$

We assert for the IGLS space $G_{\mu,\nu;a,b}^{(Q)}$:

$$\alpha(G_{\mu,\nu;a,b}^{(Q)}) = \frac{1}{b}, \quad (4.2)$$

$$\beta(G_{\mu,\nu;a,b}^{(Q)}) = \frac{1}{a}, \quad (4.3)$$

Proof. We can suppose without loss of generality that $b < \infty$ and that $\nu\{[a, b]\} = 1$.

It is sufficient also to investigate only upper Boyd's index $\beta(G_{\mu,\nu;a,b}^{(Q)})$; the case of lower index $\alpha(G_{\mu,\nu;a,b}^{(Q)})$ is investigated analogously.

A. Upper bound. Let $f = f(t)$ be arbitrary positive function from the space $G_{\mu,\nu;a,b}^{(Q)}$. We have consequently:

$$|\sigma_s[f]|_p^p = \int_0^\infty |f(t/s)|^p dt = \int_0^\infty s \int_0^\infty |f(t)|^p dt = s|f|_p^p;$$

$$|\sigma_s[f]|_p \leq s^{1/a} |f|_p, \quad s > 2;$$

$$\|\sigma_s\| \leq s^{1/a}, \quad \beta(G_{\mu,\nu;a,b}^{(Q)}) \leq 1/a.$$

B. Lower bound. We set for simplicity $Q = 1$. We have for at the same continuous positive function f from the space $G_{\mu,\nu;a,b}^{(Q)}$ and each "small" positive number ϵ :

$$I := \int_a^b s^{1/p} |f|_p dp \geq s^{1/(a+\epsilon)} \int_a^{a+\epsilon} |f|_p dp, \quad s > 2;$$

Therefore

$$\frac{\|\sigma_s\|}{\log s} \geq \frac{1}{a+\epsilon} - \frac{C(\epsilon)}{|\log s|};$$

$$\overline{\lim}_{s \rightarrow \infty} \frac{\|\sigma_s\|}{\log s} \geq \frac{1}{a}.$$

Remark 4.1. At the same result with at the same proof is true for arbitrary $Y = CGLS(T, \mu; a, b; <> \text{space})$.

But for the $G_{\mu,\nu;a,b}^{(Q)}$ space with $T = R_+$ for the Boyd's index our result may be refined. Namely, let the function $f, f : T \in R$ be such that the function $p \rightarrow |f|_p$ is positive, bilateral bounded and is continuous in some neighborhood of the point $p = a : p \in (a, a + \epsilon)$. Then we conclude as $s \rightarrow \infty, s > 3$ using the saddle-point method:

$$\|\sigma_s\|^Q \geq C \int_a^b s^{Q/p} |f|_p^Q dp \geq C_2 \int_a^b s^{Q/p} dp \geq C_3 C_4^Q \frac{s^{Q/a}}{\log s};$$

$$\|\sigma_s\| \geq C_5 \frac{s^{1/a}}{|\log s|^{1/Q}}.$$

Therefore, we conclude for sufficiently greatest values s :

$$\frac{1}{a} - \frac{\log |\log s|}{Q |\log s|} - \frac{C_6}{|\log s|} \leq \frac{\log \|\sigma_s\|}{|\log s|} \leq \frac{1}{a} \quad (4.4)$$

and we find analogously in the case when $b < \infty$ for the smallest values $s, s \in (0, e^{-2e})$:

$$\frac{1}{b} \leq \frac{\log \|\sigma_s\|}{|\log s|} \leq \frac{1}{b} + \frac{\log |\log s|}{Q|\log s|} + \frac{C_7}{|\log s|}. \quad (4.5)$$

5 Fundamental function

Recall that the fundamental function $\phi = \phi_Y(\delta)$, $\delta \in (0, \infty)$ for arbitrary r.i. space Y with norm $\|\cdot\|_Y$ is defined by formula

$$\phi_Y(\delta) = \|I(A)\|_Y. \quad (5.1)$$

In this section we investigate some properties of the fundamental function for the Integral Grand Lebesgue Spaces (IGLS) $Y = G_{\mu, \nu; a, b}^{(Q)}$.

Note that in the considered case

$$\phi_Y^Q(\delta) = \phi_{G_{\mu, \nu; [a, b]}^{(Q)}}^Q(\delta) = \int_{[a, b]} \delta^{Q/p} \nu(dp), \quad (5.2)$$

or after substitution $\delta = \exp(\lambda/Q)$, $\lambda \in (-\infty, \infty)$:

$$\zeta_Y(\lambda) = \zeta(\lambda) = \zeta_Y^{(Q)}(\lambda) \stackrel{\text{def}}{=} \phi_Y^Q(e^{\lambda/Q}) = \int_{[a, b]} e^{\lambda/p} \nu(dp). \quad (5.3)$$

Theorem 5.1. Let $\nu([a, b]) < \infty$.

A. The function $\zeta_Y(\lambda)$ is absolutely monotonic on the set $\lambda \in (-\infty, \infty)$:

$$\zeta^{(k)}(\lambda) > 0, \quad k = 0, 1, 2, \dots \quad (5.4)$$

and such that

$$C_1 e^{\lambda/b} \leq \zeta(\lambda) \leq C_2 e^{\lambda/a}. \quad (5.5)$$

We denote further the class of a functions satisfying the conditions (5.4) and (5.5) as $AM = AM(a, b)$. For instance, any function from the class $AM(a, b)$ is infinitely differentiable on the whole axis R^1 .

B. Conversely, if the function $\lambda \rightarrow \zeta(\lambda)$, defined on the set $\lambda \in (-\infty, \infty)$ satisfies the conditions (5.4) and (5.5), then there exists a finite Borelian measure on the closed interval $[a, b]$ for which

$$\zeta(\lambda) = \int_{[a, b]} e^{\lambda/p} \nu(dp). \quad (5.6)$$

Proof. The proof of the assertion **A** is very simple. Indeed:

$$\zeta^{(k)}(\lambda) = \int_{[a, b]} p^{-k} e^{\lambda/p} \nu(dp) > 0;$$

$$\zeta(\lambda) \leq e^{\lambda/a} \nu([a, b]) = C_2 e^{\lambda/a},$$

$$\zeta(\lambda) \geq e^{\lambda/b} \nu([a, b]) = C_1 e^{\lambda/b}.$$

Proof of the assertion **B**. Let the inequality (5.4) holds. From the theorem of S.N. Bernstein follows that there exists a finite measure $\tilde{\nu}$ on the real axis $(-\infty, \infty)$ for which

$$\zeta(\lambda) = \int_{[0, \infty)} e^{\lambda x} \tilde{\nu}(dx).$$

We conclude by virtue of the inequalities (5.5) that the support of the measure $\tilde{\nu}$ contained in the set $x \in [1/b, 1/a]$:

$$\zeta(\lambda) = \int_{[1/b, 1/a]} e^{\lambda x} \tilde{\nu}(dx). \quad (5.7)$$

The assertion **B** of theorem (5.1) follows from (5.7) after substitution $x = 1/p$.

Remark 5.1. Note in additional that the function $\zeta(\lambda)$ satisfies the following inequalities:

$$b^{-k} \zeta^{(k)}(\lambda) \leq \zeta(\lambda) \leq a^{-k} \zeta^{(k)}(\lambda), \quad \lambda \in R, k = 0, 1, 2, \dots$$

We continue the investigation of the properties of the fundamental function for the spaces $Y = G_{\mu, \nu; a, b}^{(Q)}$, $a < b < \infty$.

Theorem 5.2. Suppose the measure $\nu(\cdot)$ is absolutely continuous relative Lebesgue measure:

$$\nu(B) = \int_B h(p) dp,$$

where the function $h(p)$ is non-negative, integrable and continuous at the points $p = a$ and $p = b$ such that $h(a) > 0, h(b) > 0$.

Then as $\delta \rightarrow \infty$

$$\phi_Y(\delta) \sim \left[\frac{h(a)}{Q \log \delta} \right]^{1/Q} \cdot \delta^{1/a}, \quad (5.8)$$

and as $\delta \rightarrow 0+$

$$\phi_Y(\delta) \sim \left[\frac{h(b)}{Q \log \delta} \right]^{1/Q} \cdot \delta^{1/b}. \quad (5.9)$$

Proof follows immediately from the theory of saddle-point method, see [7], chapter 2, section 4.

Example 5.1. Let

$$\nu(B) = \int_B p^{-2} dp,$$

then

$$\phi_Y^Q(\delta) = \int_a^b \delta^{Q/p} \frac{dp}{p^2} = \frac{\delta^{Q/a} - \delta^{Q/b}}{Q \log \delta},$$

therefore

$$\phi_Y(\delta) = \left[\frac{\delta^{Q/a} - \delta^{Q/b}}{Q \log \delta} \right]^{1/Q}.$$

6 Conjugate and associate spaces

Since the CGLS space Y has the ACN property, the conjugate space Y^* coincides with associate space Y' . Therefore, every continuous (bounded) linear functional $l, l : Y \rightarrow R$ has a view

$$l(f) = l_g(f) = \int_T f(t)g(t)\mu(dt), \quad (6.1)$$

where $g : T \rightarrow R$ is some measurable function for which

$$\|g\|_{Y^*} = \|g\|_{Y'} = \sup_{f: \|f\|_Y=1} \left| \int_T f(t)g(t)\mu(dt) \right| < \infty. \quad (6.2)$$

But the expression (6.2) is very hard to calculate. We give now a simple upper estimation for the norm $\|g\|_{Y^*}$.

Let us consider the Integral Grand Lebesgue Space (IGLS) $G_{\mu,\nu}^{(Q)}$. We denote as usually $Q' = Q/(Q-1)$, $p' = p/(p-1)$, $Q, p > 1$ and define

$$\|g\|_{Q'} = \|g\|_{Q',\nu} = \left[\int_a^b |g|_{p'}^{Q'} \nu(dp) \right]^{1/Q'}.$$

Theorem 6.1. We assert for the space $Y = G_{\mu,\nu}^{(Q)}$:

$$\|g\|_{Y^*} \leq \|g\|_{Q'}. \quad (6.3)$$

Proof. As long as the measure ν is sigma-finite, we can and will suppose $\nu((a,b)) = 1$. We obtain using Hölder's inequality for the representation (6.1):

$$|l_g(f)| = \left| \int_T f(t)g(t)\mu(dt) \right| \leq |f|_p |g|_{p'};$$

therefore

$$\begin{aligned} |l_g(f)| &\leq \inf_{p \in (a,b)} [|f|_p |g|_{p'}] \leq \int_a^b |f|_p |g|_{p'} \nu(dp) \leq \\ &\left[\int_a^b |f|_p^Q \nu(dp) \right]^{1/Q} \cdot \left[\int_a^b |g|_{p'}^{Q'} \nu(dp) \right]^{1/Q'} = \|f\|_Y \cdot \|g\|_{Q'}, \end{aligned}$$

Q.E.D.

Note that the estimation (6.3) of theorem 6.1 is exact when the measure ν is Dirac measure and is exact up to multiplicative constant for the pure discrete measure ν with finite support.

But in general case the expression for $\|g\|_{Q'}$ does not represent the general form for linear functional over the space $G_{\mu,\nu}^{(Q)} = G_{\mu,\nu}^{(Q)}(a, b)$. Let us consider correspondent example.

Theorem 6.2. Suppose the measure ν is absolutely continuous over Lebesgue measure with positive continuous differentiable Radon-Nikodim derivative $h(p)$:

$$\nu(B) = \int_B h(p) dp.$$

where the function $h(p)$ is non-negative, integrable and continuous at the points $p = a$ or at the point $p = b$ and such that $h(a) > 0, h(b) > 0$. Then the Integral Grand Lebesgue Space (IGLS) $G_{\mu,\nu}^{(Q')}(b/(b-1), a/(a-1)), a > 1, b < \infty$ does not coincide with the dual (associate) space to the space $G_{\mu,\nu}^{(Q)}(a, b)$.

Proof. Assume conversely. i.e. that

$$G_{\mu,\nu}^{(Q')}(b/(b-1), a/(a-1)) = G_{\mu,\nu}^{(Q)}(a, b).$$

It is known (see [4]), chapter 1,2 that

$$\phi_Y(\delta) \cdot \phi_{Y'}(\delta) = \delta, \quad \delta \in (0, \infty).$$

The last equality contradict the proposition of theorem 5.2 when $\delta \rightarrow \infty$ or as $\delta \rightarrow 0 +$.

7 Relations with another r.i. spaces.

We intend to prove in this section that the classical r.i. spaces: Lorentz, Orlicz, Marzinkiewicz, Grand Lebesgue Spaces does not coincide or equivalent to the considered in this article Composed Grand Lebesgue Spaces.

Note first of all that the *exponential* Orlicz's and Grand Lebesgue Spaces are under simple conditions not separable [19], [39], in contradiction to the properties of CGLS spaces.

We introduce *in this section* the following equivalence relation \asymp between a two positive functions $g_1(\lambda), g_2(\lambda)$ defined on the whole real axis:

$$g_1(\cdot) \asymp g_2(\cdot) \Leftrightarrow \exists C_1, C_2 = \text{const} > 0, C_1 \leq C_2, \quad C_1 \leq \frac{g_1(\lambda)}{g_2(\lambda)} \leq C_2. \quad (7.0)$$

Now, let $X, \|\cdot\|_X$ be r.i. space over our measurable triple (T, \mathcal{A}, μ) ; denote by $\phi_X(\delta)$ its fundamental function and define for some $Q = \text{const} \geq 1$ and for arbitrary $\lambda \in R$ the following function:

$$\tau_X(\lambda) = \tau_X^{(Q)}(\lambda) := \phi_X^Q(e^{\lambda/Q}). \quad (7.1)$$

It follows from the properties of the fundamental functions of CGLS spaces (theorem 5.1) the following result.

Theorem 7.1.

A. If

$$\tau_X^{(Q)}(\cdot) \notin \cup_{(a,b): 1 \leq a < b < \infty} \{AM(a, b)\}, \quad (7.2)$$

then the r.i. space X does not coincide with arbitrary $G_{\mu,\nu}^{(Q)}(a, b)$ space.

2. If for fixed r.i. space X and for all the values (a, b) , Q its function $\tau_X^{(Q)}(\cdot)$ is not equivalent in the relation \asymp to arbitrary function from the class $AM(a, b)$, then the space X is not equivalent (in the sense of Banach space equivalency) to arbitrary $G_{\mu,\nu}^{(Q)}(a, b)$ space.

For instance, the classical r.i. Lorentz and Marzinkiewicz spaces, see [17], chapter 3. may have the non-smooth fundamental function; therefore this spaces does not coincide with arbitrary $G_{\mu,\nu}^{(Q)}(a, b)$ space.

8 Convergence and compactness

As we know, the space $G_{\mu,\nu}^{(Q)}(a, b)$ obeys the Absolutely Continuous Norm property. We conclude consequently:

Theorem 8.1. Let $F = \{f_\alpha\} \subset G_{\mu,\nu}^{(Q)}(a, b)$, $\alpha \in A$, where A is arbitrary set of indices be any subset of the space $G_{\mu,\nu}^{(Q)}(a, b)$.

The set F is compact set in this space iff it is bounded, closed and obeys the Uniform Absolutely Continuous Norm property.

Proof follows immediately from the theory of a rearrangement invariant spaces, see [4], chapter 2; [14], chapter 4, section 3.

A slight consequence:

Theorem 8.2. The sequence $\{f_n\}$, $n = 1, 2, \dots$ of a functions from the space $G_{\mu,\nu}^{(Q)}(a, b)$ converges as $n \rightarrow \infty$ iff it converges in μ - measure and has the Uniform Absolutely Continuous Norm property.

Obviously, the convergence in measure may be replaced by convergence in some L_p , $p \in (a, b)$.

Let (T, \mathcal{A}, μ) be again some measurable space with sigma-finite non - trivial measure μ . We assume in addition that $T = \{t\}$ is homogenous compact metric space with additive operation $t \pm s$. We define the difference operator $\Theta_h[f]$, $h, t \in T$ as ordinary:

$$\Theta_h[f](t) = f(t + h) - f(t).$$

The application of Shilov's theorem give us the following result:

Theorem 8.3. Let $F = \{f_\alpha\} \subset G_{\mu,\nu}^{(Q)}(a, b)$, $\alpha \in A$, where A is arbitrary set of indices be any subset of the space $G_{\mu,\nu}^{(Q)}(a, b)$.

The set F is compact set in this space iff it is bounded, closed and

$$\limsup_{h \rightarrow 0} \sup_{f \in F} \langle \Theta_h[f_\alpha](\cdot) \rangle = 0.$$

9 Some applications: boundedness of Hilbert's transform and other operators in CGLS

Let $1 \leq a < b \leq \infty$ and let U be an operator, not necessary to be linear or sublinear, such that there exists non-zero finite "constant" $K = K(p)$, $a < p < b$ for which

$$|U[f]|_{L_p(X, \mu)} \leq K(p) |f|_{L_p(X, \mu)}, \quad f \in L_p(X, \mu). \quad (9.1)$$

The condition (9.1) is satisfied, e.g. for weight Fourier transform [3], [18], for Hilbert's transform: $a = 1, b = \infty$, $K(p) = Cp^2/(p-1)$ [45], Hardy-Littlewood maximal operator: $a = 1, b = \infty$, $K(p) = Cp/(p-1)$ [11], mean value operator

$$U[a](n) = n^{-1} \sum_{k=1}^n a(k), \quad K(p) = Cp/(p-1),$$

[11], and many others singular operators.

We will understand furthermore as the value $K(p)$ its minimal value, namely:

$$K(p) = \sup_{f \in L_p, f \neq 0} \left[\frac{|U[f]|_{L_p(X, \mu)}}{|f|_{L_p(X, \mu)}} \right]. \quad (9.2)$$

This constants is calculated in many works, see e.g. in [11], [18], [45], [47].

We introduce a new CGLS space as a set of all (measurable) functions with finite norm

$$|||g||| \stackrel{def}{=} \langle |g|_p / K(p) \rangle. \quad (9.3)$$

Theorem 9.1.

$$|||U[f]||| \leq 1 \cdot \langle f \rangle, \quad (9.4)$$

where the constant "1" in (9.4) is the best possible.

Proof. The lower bound is attained, for instance, when the measure ν is Dirac's $\delta(p - p_0)$ measure. The upper bound may be proved very simple:

$$\begin{aligned} |||U[f]||| &= \langle |U[f]|_p / K(p) \rangle \leq \langle K(p) |f|_p / K(p) \rangle = \\ &= \langle |f|_p \rangle = \langle f \rangle. \end{aligned} \quad (9.5)$$

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